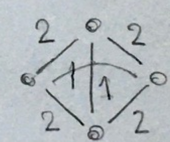
1.1 The Probabilistic Method

Rough idea: To show that an object O with certain properties exists, we
(i) define a probability space of objects, including O (usually easy), and
(ii) show that $\Pr(O) > 0$.

Def. The Ramsey number $R(k, l)$ is the smallest integer n such that in every 2-coloring of the edges of K_n with red and blue, either there is a red K_k or there is a blue K_l.

Examples:

- $R(1,1) = 1$
- $R(k,1) = 1$
- $R(k,l) = R(l,k)$
- $R(2,2) = 2$
- $R(k,2) = k$

- $R(3,3) > 4$ 

$R(3,3) = 6$ (this is not obvious).

Ramsey (1929): $R(k, l)$ is finite for any two k, l.

The exact value of Ramsey numbers is only known for small k and l.

Proposition (Lower bound on $R(k,k)$): If $\binom{n}{k} \cdot 2^{1-\binom{k}{2}} < 1$ then $R(k,k) > n$.

Proof: WTS that for such an n, there exists a 2-coloring of the edges of K_n with no monochromatic K_k.

Object

Let X be a random 2-coloring of the edges of K_n. WTS

$$\Pr_X [K_n \text{ has no monoch. } K_k \text{ under } X] > 0$$

$$\Leftrightarrow \Pr[K_n \text{ has monoch. } K_k \text{ under } X] < 1.$$

$$\Pr[K_n \text{ has monoch. } K_k \text{ under } X] = \Pr\Big[\bigcup_{K_k \subseteq K_n} K_k \text{ is monoch. under } X\Big]$$

$$\leq \sum_{K_k \subseteq K_n} \Pr[K_k \text{ is monoch. under } X]$$

$$= \binom{n}{k} \cdot 2 \cdot \left(\frac{1}{2}\right)^{\binom{k}{2}}$$

$$= \binom{n}{k} \cdot 2^{1-\binom{k}{2}}$$

$$< 1 \quad (\text{by assumption}). \ \square$$

Corollary: $R(k,k) > \lfloor 2^{k/2} \rfloor$ for all $k \geq 3$.

Proof: WTS that $n = \lfloor 2^{k/2} \rfloor$ satisfies the assumption of the previous prop

$$\binom{n}{k} \cdot 2^{1-\binom{k}{2}} \leq \frac{n^k}{k!} \cdot 2^{1-k(k-1)/2} \qquad \left(\text{since } \binom{n}{k} \leq \frac{n^k}{k!}\right)$$

$$= \frac{n^k}{k!} \cdot 2^{1+\frac{k}{2}-\frac{k^2}{2}}$$

$$= \frac{n^k}{2^{k^2/2}} \cdot \frac{2^{1+\frac{k}{2}}}{k!}$$

$$\leq 1 \cdot \frac{2^{1+k/2}}{k!} \qquad \left(\text{since } n = \lfloor 2^{k/2} \rfloor \Rightarrow n^k \leq 2^{k^2/2}.\right)$$

$$< 1 \quad \text{for } k \geq 3. \qquad \square$$

Is probability essential in this proof? In this case, we could've come up with a combinatorial proof: The number of 2-colorings X that yield a monochromatic K_k is

$$\leq \underbrace{\binom{n}{k}}_{\substack{\text{Select} \\ \text{monochrom.} \\ K_k}} \cdot \underbrace{2}_{\substack{\text{Can color} \\ \text{it} \\ \text{2 ways}}} \cdot \underbrace{2^{\binom{n}{2}-\binom{k}{2}}}_{\substack{\text{Color all} \\ \text{other} \\ \text{edges}}} = \binom{n}{k} \cdot 2^{1-\binom{k}{2}} \cdot 2^{\binom{n}{2}}.$$

By assumption, this is $< 2^{\binom{n}{2}}$, which is the # of 2-colorings.

In other cases, probability is essential. Namely, when we use probabilistic tools, like the second moment method or Lovász Local Lemma.

Obs: We can turn the proof into an efficient Monte Carlo algorithm to compute an edge 2-coloring X certifying $R(k,k) > n$, for $n = \lfloor 2^{k/2} \rfloor$. We have shown.

$$\Pr[K_n \text{ has monoch. } K_k \text{ under } X] \leq \underbrace{\frac{2^{1+\frac{k}{2}}}{k!}}_{\text{Tiny fraction!}} = o(1)$$

Algorithm: Sample random X and return.

The answer X is good w.p. $1-o(1)$.