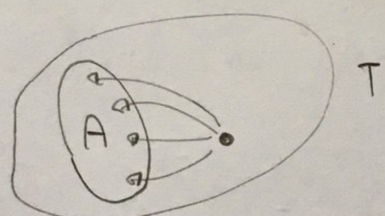
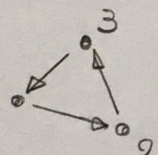
1.2 Graph Theory

Def: A tournament on a set V of n players is an orientation $T=(V,E)$ of the complete graph on V, such that for all $x \neq y \in V$, either $xy \in E$ or $yx \in E$ but not both.

We say that x beats y if $xy \in E$.

We say that a tournament has property S_k if for every $A \subseteq V$ of k players, $\exists$ player that beats everyone in A.



Example: $T =$ 

has S_1 but not S_2.

Obs: S_k only makes sense when $n > k$.

If $n < k$, S_k holds vacuously.

If $n = k$, S_k doesn't hold.

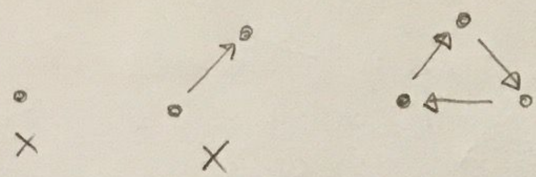
Question: Given k, what is the smallest size of a tournament with S_k?

Is there one?

Def: $f(k) = \min\{n > k \mid \exists$ tournament on n vertices with $S_k\}$

Examples:

$f(1) = 3$



$f(2) = 7$

$f(2) \geq 7 \Leftrightarrow$ every tournament on ≤ 6 vertices doesn't have S_k. (hard)

$f(2) \leq 7 \Leftrightarrow$ some tournament on 7 vertices has S_k (easy)

Question (Schütte): Is $f(k) < \infty$ for all k?

Theorem [Erdős '63]: If $\binom{n}{k}(1-2^{-k})^{n-k} < 1$ then $f(k) \leq n$.

Proof:

ETS that $\exists$ tournament on n players that has S_k:

Let T be a random n-player tournament. WTS $\Pr_T[T$ doesn't have $S_k] < 1$

for each $x \neq y$, $x \to y$ w.p. $\frac{1}{2}$ or $y \to x$ oth.

$$\Pr[T \text{ doesn't have } S_k] = \Pr[\exists A \subseteq V, |A| = k, \text{ s.t. no player beats } A]$$

$$= \sum_{\substack{A \subseteq V \\ |A| = k}} \Pr[\text{no player beats } A]$$

$$= \sum_{A} \Pr[\forall x \in V \setminus A, \ x \text{ doesn't beat } A]$$

$$= \sum_{A} \prod_{x \in V \setminus A} \Pr[x \text{ doesn't beat } A]$$

$$= \sum_A \prod_x \left(1 - \Pr[x \text{ beats } A]\right)$$

$$= \sum_A \prod_x (1 - 2^{-k})$$

$$= \binom{n}{k}(1-2^{-k})^{n-k}$$

$$< 1 \quad (\text{by assumption}) \quad \square$$

<u>Corollary</u>: $f(k) = O(k^2 2^k)$.

Proof:

WTS $n = O(k^2 2^k)$ satisfies $\binom{n}{k}(1-2^{-k})^{n-k} < 1$.

$$\binom{n}{k} \cdot (1-2^{-k})^{n-k} < \underbrace{\left(\frac{en}{k}\right)^k}_{\substack{\text{larger} \\ \text{as } n \to \infty \\ (\approx n^k)}} \cdot \underbrace{\exp\left(-\frac{n-k}{2^k}\right)}_{\substack{\text{smaller} \\ \text{as } n \to \infty \\ (\approx e^{-n})}} \quad \left(\text{as } \binom{x}{y} < \left(\frac{ex}{y}\right)^y \text{ and } (1+x) \leq e^x \right)$$

$$= \exp\left(k + k \ln \frac{n}{k} - \frac{n-k}{2^k}\right)$$

$n = \Theta(2^k \cdot k^2)$ gives:

$$= \exp\left(k + k \underbrace{\ln(\Theta(2^k \cdot k))}_{\Theta(k)} - \Theta(k^2)\right) = \exp(-\Theta(k^2)).$$

$\square$

Theorem [Szekeres]: $f(k) \geq \Omega(k \cdot 2^k)$.