Dominating set

Def: A dominating set of an undirected graph $G = (V, E)$ is a set $U \subseteq V$ such that every $x \in V \setminus U$ has at least one neighbor in U.

Obs: Minimum dominating set is an NP-complete problem.

Theorem: Let $G = (V, E)$ be a graph on n vertices, with minimum degree $d > 1$. Then G has a dominating set of size at most $\frac{n(1 + \ln(d+1))}{d+1}$ verts.

Proof:

Let X be a random subset of vertices such that each $x \in V$ is added to X w.p. p, p TBD. Let $Y = Y_X$ be the (random) set of all vertices in $V \setminus X$ that don't have any neighbor in X. Let $U = X \dot\cup Y$.

This is a dominating set.

$$E[|U|] = E[|X|] + E[|Y|]$$

$$E[|X|] = np$$

$$E[|Y|] = \sum_{v \in V} E[\mathbb{1}\{v \in V \setminus X,\ v \text{ doesn't have a neighbor in } X\}]$$

$$= \sum_{v \in V} \Pr[v \in V \setminus X,\ v \text{ doesn't have a neighbor in } X].$$

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$$= \sum_v \Pr[v \notin X] \cdot \Pr[v \text{ doesn't have neighbor in } X]$$

(as these events depend on different vertices)

$$= \sum_v (1-p)\cdot(1-p)^{d(v)}$$

$$= \sum_v (1-p)^{1+d(v)}$$

$$\leq \sum_v (1-p)^{1+\delta} \quad (\text{as } d(v) \geq \delta \text{ and } 1-p \in [0,1])$$

$$= n(1-p)^{\delta+1}.$$

Thus $E[|U|] = E[|X|+|Y|] \leq np + n(1-p)^{\delta+1}$.

$\Rightarrow \exists$ a choice of X such that U has $\leq np + n(1-p)^{\delta+1}$ vertices.

(In general, $\underset{r}{E}[Z] \leq \alpha \Rightarrow$ there is a choice of the randomness r that makes Z take a value $\leq \alpha$.)

To conclude the proof, we find p that minimizes $np + n(1-p)^{\delta+1}$.

$$\leq np + ne^{-p(\delta+1)} = n\left(p + e^{-p(\delta+1)}\right).$$

$$f(p) = p + e^{-p(\delta+1)}$$

$$f'(p) = 1 + e^{-p(\delta+1)} \cdot (-(\delta+1)).$$

$$f'(p) = 0 \Leftrightarrow e^{p(\delta+1)} = \delta+1 \Leftrightarrow p = \frac{\ln(1+\delta)}{1+\delta}.$$

For $p = \ln(\delta+1)/_{\delta+1}$ we get

$$n \cdot (p + e^{-p(\delta+1)}) = n\left(\frac{\ln(\delta+1)}{\delta+1} + \frac{1}{\delta+1}\right)$$

$$= \frac{n(1+\ln(\delta+1))}{\delta+1} . \square$$

There's a way to achieve the same bound with a simple deterministic greedy algorithm:

$U \leftarrow \emptyset$

while U doesn't dominate V:

 choose $v \in V \setminus U$ that is adjacent to the largest number of vertices in V that are still not dominated

 $U \leftarrow U \cup \{v\}$

Analysis:

Let V_i be the set of vertices not dominated after the ith iteration. $V_0 = V$. Define U_i to be U in the ith iteration, and v_i the chosen vertex.

Obs: $E(U_i, V_i) = \emptyset$.