1.3 Combinatorics

Def: A hypergraph $H = (V, E)$, where V is a finite set (the vertices), and E is a family of subsets of V (the edges). It is k-uniform if every edge contains exactly k vertices.

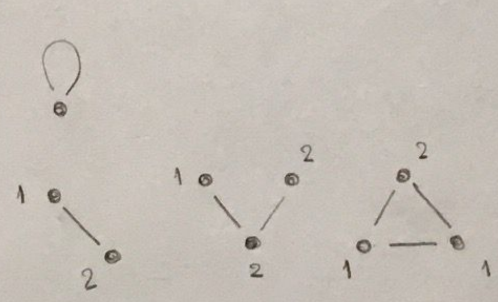
Def: H has property $\mathcal{B}$, or it's 2-colorable, if there is a 2-coloring of V such that no edge is monochromatic.

For $k=2$ (i.e., graphs) 2-colorable is a very stringent property: There are 4 ways of coloring an edge with 2 colors, and only 2 of them satisfy $\mathcal{B}$. For large k, it's much easier to do it: Only 2 out of 2^k colorings do not satisfy $\mathcal{B}$.

Def: $m(k) = \min\{ m \mid \exists\ k\text{-uniform hypergraph on } m \text{ edges that doesn't have prop. } \mathcal{B} \}$

$m(1) = 1$

$m(2) = 3$



Proposition [Erdős '63]: Every k-uniform hypergraph with $< 2^{k-1}$ edges has property $\mathcal{B}$. I.e. $m(k) \geq 2^{k-1}$.

Proof: Let $H = (V, E)$ be a k-uniform hypergraph on $m < 2^{k-1}$ edges.
Let φ be a random coloring of V with 2 colors. WTS

$$\Pr[H \text{ has no monochromatic edge under } \varphi] > 0$$

$$\Leftrightarrow \Pr[H \text{ has monochromatic} \ldots] < 1$$

$$\begin{aligned}
\Pr[H \text{ has monochromatic} ..] &= \Pr[\exists e \in E : e \text{ monochromatic}] \\
&\leq \sum_{e \in E} \Pr[e \text{ monochromatic}] \\
&= m \cdot 2 \cdot 2^{-k} \\
&= m / 2^{k-1} \\
&< 1 \quad (\text{by assumption}). \qquad \square
\end{aligned}$$

Cherkashin & Kozik 2015: $m(k) \geq \Omega\left(\left(\frac{k}{\ln k}\right)^{1/2} \cdot 2^k\right)$.

Theorem [Erdős '64]: $m(k) \leq O(k^2 2^k)$

Proof: WTS $\exists$ k-uniform hypergraph with $O(k^2 2^k)$ edges doesn't have property $\mathcal{B}$.

Fix V with n vertices, n TBD. Fix m TBD. Let $E = \{S_1, \ldots, S_m\}$ be a random set of hyperedges with $|S_i| = k$. Let $H = (V, E)$. WTS

$$\Pr[H \text{ doesn't have property } \mathcal{B}] > 0$$

$$\Leftrightarrow \Pr[H \text{ has property } \mathcal{B}] < 1.$$

$$\Pr[H \text{ has property } B] = \Pr[\exists\, \varphi \text{ 2-coloring of } V \text{ such that no } e \in E \text{ is monochromatic}]$$

$$\leq \sum_{\varphi} \Pr[\text{no } e \text{ is monochromatic under } \varphi]$$

$$= \sum_{\varphi} \prod_{i=1}^{m} \Pr_{S_i}[S_i \text{ is not monochromatic under } \varphi]$$

(all edges are independent)

$$= \sum_{\varphi} \prod_{i=1}^{m} (1 - \underbrace{\Pr_{S_i}[S_i \text{ is monoch. under } \varphi]}_{p})$$

$$= 2^n \cdot (1-p)^m < 2^n e^{-pm} \quad (*)$$

$(*)$: If $m = \frac{n}{p} \ln 2$, this is < 1 and we're done.

Fix an arbitrary coloring φ. Let a be the number of red vertices and b the number of blue vertices. Then

$$p = \frac{\binom{a}{k}}{\binom{n}{k}} + \frac{\binom{b}{k}}{\binom{n}{k}} = \frac{\binom{a}{k} + \binom{b}{k}}{\binom{n}{k}}, \qquad a + b = n$$

For what a, b is p minimized?

Claim: $f(x) = \binom{x}{k}$ is convex for $x \geq k$.

Proof: $f(x)$ is just a polynomial of degree k with roots $0, 1, \dots, k-1$ and main coefficient > 0.

$$f(x) = \frac{x!}{k!\,(x-k)!} = \frac{1}{k!} (x-(k-1))(x-(k-2)) \cdots x \quad . \quad \square$$

Thus if $k \leq x, y \leq n$ and $x+y=n$, $f(x)+f(y)$ is minimized for $x=y=\frac{n}{2}$. (provided $k \leq \frac{n}{2}$). Thus

$$p \geq \frac{\binom{n/2}{k}+\binom{n/2}{k}}{\binom{n}{k}} = \frac{2\binom{n/2}{k}}{\binom{n}{k}}$$

$$= 2\,\frac{(n/2)!/\cancel{k!}\,(n/2-k)!}{n!/\cancel{k!}\,(n-k)!}$$

$$= 2\,\frac{(n/2-k+1)\cdots n/2}{(n-k+1)\cdots n}$$

$$= \frac{2}{2^k}\cdot\frac{(n-2(k-1))\cdot(n-2(k-2))\cdots n}{(n-(k-1))\,(n-(k-2))\cdots n}$$

$$= \frac{1}{2^{k-1}}\cdot\prod_{i=0}^{k-1}\frac{n-2i}{n-i}$$

$$= \frac{1}{2^{k-1}}\cdot\prod_{i=0}^{k-1}\left(1-\frac{i}{n-i}\right)$$

$$\geq \frac{1}{2^{k-1}}\prod_{i=0}^{k-1}\exp\left(-\frac{2i}{n-i}\right)$$

assuming $n \gg k$ and using ($1-x \geq e^{-2x}$ for $x\in[0,\frac{1}{2}]$)

$$= \frac{1}{2^{k-1}}\exp\left(-2\sum_{i=0}^{k-1} i/n-i\right)$$

$$= \frac{1}{2^{k-1}}\cdot\exp\left(-\Theta\left(\frac{k^2}{n}\right)\right)$$

(as $n \gg k$)

$$\geq \Omega\left(\frac{1}{2^k}\right)$$

(maximized for $n=\Theta(k^2)$)

Thus $m = \frac{n}{p}\ln 2 = O(n\,2^k) = O(k^2 2^k)$. $\square$